If, now, we put

$$
\begin{aligned}
l_{\prime} &= k \sin\varphi \sin\eta_0, & m_{\prime} &= -\sin\varphi\cos\eta_0, \\
l_{\prime\prime} &= -k\sin\eta_0, & m_{\prime\prime} &= \cos\eta_0, \\
l_{\prime\prime\prime} &= k\cos\varphi\cos\eta_0, & m_{\prime\prime\prime} &= \cos\varphi\sin\eta_0;
\end{aligned}
$$

we may readily obtain the following general equations:—

$$
\begin{aligned}
\frac{r}{a} &= 1 + l_{\prime}\lambda(z) + m_{\prime}\nu(z), \\
\frac{r}{a}\cos f &= l_{\prime\prime}\lambda(z) + m_{\prime\prime}\nu(z) - e, \\
\frac{r}{a}\sin f &= l_{\prime\prime\prime}\lambda(z) + m_{\prime\prime\prime}\nu(z), \\
\frac{dg}{dz} &= k\frac{r}{a}\mu(z);
\end{aligned}
$$

where, as may be seen at a glance, $e = \sin\varphi$ is the eccentricity, a the semi major axis, r the radius vector, f the true anomaly, and g the mean anomaly.

If the intervals between the different values of η_0 be the same, k will be the same in all parts of the orbit, and a single development only will be needed of the quantities $\lambda(z)$, $\mu(z)$, and $\nu(z)$.

ON THE DESIGN OF STEPPED PULLEYS FOR LATHE GEARS.

By Prof. Wm. M. Thornton, University of Virginia.

The problem to be solved in this case is the determination of the diameters of a set of pairs of pulleys which will transmit the motion of the shaft to the lathe spindle with given angular velocity ratios by a belt of constant length. The fundamental formulæ for the solution of this problem are easily written down. We have

$$
\begin{aligned}
L &= (\pi + 2\theta)R + (\pi - 2\theta)r + 2c\cos\theta, \\
\sin\theta &= \frac{R-r}{c}, \\
n &= \frac{R}{r};
\end{aligned}
$$

where L is the constant length of belt,

c the distance between the pulley centres,

n the given angular velocity ratio,

R the radius of the larger pulley,

r the radius of the smaller pulley,

θ the angle between the line of centres and the straight belt.

The data are L, c, n; R, r are required.

The solution of this group of equations is effected in the present paper by using the auxiliary quantity $x = \sin\theta$ to deduce a relation between the known ratio $\frac{L}{c}$, the given velocity ratio n, and x. If x be found from this relation we can obtain R and r at once from the formulæ

$$r = \frac{x}{n-1}c, \quad R = \frac{nx}{n-1}c.$$

To carry out this solution we reduce the expression for $\frac{L}{c}$ to the form

$$\frac{L}{c} = 2(\theta\sin\theta + \cos\theta) + \frac{n+1}{n-1}\pi\sin\theta,$$

and in it put

$$\sin\theta = x,$$
$$\theta = x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \frac{5}{112}x^7 + \ldots,$$
$$\cos\theta = 1 - \frac{1}{2}x^2 - \frac{1}{8}x^4 - \frac{1}{16}x^6 - \frac{5}{128}x^8 - \ldots,$$
$$L = 2c + 2\pi R_0,$$
$$\rho = \frac{n-1}{n+1};$$

R_0 representing the radius of either of the pair of equal pulleys. After suitable reductions are made we find

$$x = 2\rho\frac{R_0}{c} - \frac{\rho}{\pi}x^2\left(1 + \frac{1}{12}x^2 + \frac{1}{40}x^4 + \frac{5}{448}x^6 + \ldots\right).$$

If we neglect the variable terms in the bracket and put for x on the right the first approximation $2\rho\frac{R_0}{c}$, we get the quite close approximation

$$\xi = 2\rho\frac{R_0}{c} - \frac{\rho}{\pi}\left(2\rho\frac{R_0}{c}\right)^2.$$

Putting ξ for x in the parentheses we get a still closer approximation, which may in turn be used as the basis for a closer one; and so on, until the primitive and the derived values of x agree to the requsite number of decimal places.

For example, take the case in which $c = 120'$ and $R_0 = 12'$. We find

$$L = 240 + 24\pi = 315{,}40,$$

and for the radii of the pulleys which must be used with the same belt to communicate the angular velocity ratio 3:2 we get

$$\begin{aligned}
\xi &= 2 \cdot \frac{1}{5} \cdot \frac{1}{10} - 0{,}3183 \cdot \frac{1}{5} \cdot \left(\frac{1}{25}\right)^2, \\
&= 0{,}0399, \\
\therefore\ x &= 0{,}0399, \\
r &= 9{,}58, \\
R &= 14{,}37.
\end{aligned}$$

The error committed in using ξ for x is obviously less than

$$\frac{4}{3\pi}\rho^5\left(\frac{R_0}{c}\right)^4,$$

and may therefore be disregarded if the value of this expression is less than a half unit of the last place retained in the value of x. Thus in the example above this error is less than $136 : 10^{10}$, so that if it were desired we could carry the value of ξ to seven decimal places and it would agree to the last place inclusive with the value of x. We should get thus

$$x = 0{,}0398981.$$

Such closeness of approximation of course far exceeds the capacities of the machine shop. It will be found that in almost every case of actual practice the approximation

$$x = 2\rho\frac{R_0}{c} - \frac{\rho}{\pi}\left(2\rho\frac{R_0}{c}\right)^2$$

will furnish values of r and R exact to the nearest hundredth of an inch.

A convenient method of determining with rapidity and with an exactness sufficient for practical purposes the value of x corresponding to given values of ρ and $\frac{R_0}{c}$ is to construct to a large scale the curve

$$y = x^2 + \frac{1}{12}x^4 + \frac{1}{40}x^6 + \frac{5}{448}x^8 + \ldots,$$

and mark its point of intersection with the straight line

$$\frac{x}{\rho}+\frac{y}{\pi}=2\,\frac{R_0}{c}.$$

The abscissa of this point is the required value of x and R, r are then found as above. The curve can be constructed once for all and measures can afterward be taken directly from the sheet, the straight line being drawn in by its axial intercepts.

The methods heretofore in use for the design of these pulleys proceed by assuming values for the difference $R - r$ instead of for the ratio $R:r$. Sang, for example, calculates the diameter of a circle equal in length to the belt $\left(B = \frac{L}{\pi c}\right)$, the distance between the centres of the pulleys being the linear unit, and using as argument the difference $R - r = x$ computes the differences

$$B - 2r = \frac{2}{\pi}\left\{1 + \frac{1}{2}x^2 + \frac{1}{24}x^4 + \ldots\right\} + x,$$

$$B - 2R = \frac{2}{\pi}\left\{1 + \frac{1}{2}x^2 + \frac{1}{24}x^4 + \ldots\right\} - x.$$

These differences being tabulated for equidistant values of x, we get from the table by interpolation $B - 2r, B - 2R$; whence r, R.* Culmann's method is based on the equivalent formulæ

$$2r = B - \frac{2}{\pi}\left[x \sin^{-1}x + \sqrt{(1 - x^2)}\right] - x,$$

$$2R = B - \frac{2}{\pi}\left[x \sin^{-1}x + \sqrt{(1 - x^2)}\right] + x;$$

which are used as the foundation of a grapical process for finding from a given x the values of $2R - x$ and $2r + x$.† The objection to all such methods is the fact that the determination ought to be based in all cases on the ratio of the radii.

*Spon's *Encyclopedia*: Belts.

†Cf. Moll u. Reuleaux, *Constructionslehre*, p. 321.